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Two Linearization Procedures for the Boltzmann Equation in a $k = 0$ Robertson–Walker Space-Time

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It is the aim of this paper to describe two different linearization procedures for the Boltzmann equation in a $k = 0$ Robertson–Walker space-time. These procedures are discussed with a view to obtaining an asymptotic form of the Boltzmann equation for the late stages of cosmic expansion where the behavior appropriate to a nonrelativistic gas is encountered. Using the asymptotic kinetic equations, a necessary and sufficient condition is formulated under which every small perturbation of the equilibrium distribution function, either classical or relativistic, decays to zero as time goes on. The same condition can be extracted from each of two linearization procedures, and in this sense a comparison is made of these approaches which reveals mutual agreement. Also, applying an asymptotic theory of the Einstein–Boltzmann system, we show that the final state of a gas is dust (i.e., a fluid with zero temperature and pressure). Comparison with the predictions of the Eckart fluid model is briefly presented.

KEY WORDS: Boltzmann equation; Robertson–Walker space-times; linearization procedures; asymptotic kinetic equations; convergence to equilibrium; dust model; hydrodynamic description.

1. INTRODUCTION

In the relativistic kinetic theory,[1,2] we are often interested in processes close to equilibrium and for those it is reasonable to use a linearized form of the Boltzmann equation.[3] One may regard as being appropriate to equilibrium any distribution function such as to be left unaltered by colli-

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sions. However, although this definition is standard within the context of a special relativistic kinetic theory, its direct application to a fully general relativistic setting could lead to difficulties. As a typical example, it is known that the Maxwell–Jüttner distribution for a nondegenerate gas of massive particles cannot be a solution of the Boltzmann equation in an expanding universe.[4] Nevertheless, it is always possible to try to linearize the general relativistic Boltzmann equation about the Maxwell–Jüttner distribution function,[5] and in fact about any other distribution function. The only true problem one faces in that approach is to show that the resulting linearized Boltzmann equation is consistent and soluble.

The principal objective of this paper is to describe two different linearization procedures for the Boltzmann equation in a $k = 0$ Robertson–Walker space-time. The first approach linearizes the Boltzmann equation about a suitably chosen solution of Liouville's equation, denoted F, and called the classical equilibrium distribution function, and then obtains an asymptotic form of the linearized Boltzmann equation for the late stages of cosmic expansion where the behavior appropriate to a nonrelativistic gas is encountered. As to the choice of F, we propose to test the hypothesis that F is the Maxwellian distribution of microscopic momenta.[6] The second approach starts with essentially the same general program but uses the relativistic equilibrium phase density in place of F. We denote this density by $\mathbb{F}$ and call it the Maxwell–Jüttner distribution function.

Because the linearized Boltzmann equations depend intricately on time through the expansion factor R in a $k = 0$ Robertson–Walker metric, these equations are still too complicated to be readily applied to simple calculations. However, as we shall soon see, we may always effect a considerable but nontrivial simplification by carrying out the asymptotic expansion of the linearized Boltzmann equations with respect to an appropriately normalized temperature of the gas, thereby reducing them to much more tractable forms. In this context, we remark that the asymptotic expansion just mentioned is valid only for the late stages of cosmic expansion where ordinary matter behaves like a nonrelativistic gas. The additional conclusion is that, although after a long time the elementary results can be deduced from a "nonrelativistic" treatment, our asymptotic equations for perturbations are essentially new and thus have no precise analog in the classical kinetic theory.

Fixing attention on asymptotic forms of the linearized Boltzmann equations, we provide an explicit solution of the Cauchy problem. Later we shall use this solution to find a necessary and sufficient condition under which every small perturbation of the equilibrium distribution function, either classical or relativistic, decays to zero as time goes on. The same condition can be extracted from each of two linearization procedures, and in this sense a comparison is made of these approaches which reveals mutual agreement. Moreover, the equivalence of both methods can be established in other ways. Various refinements, generalizations, and cosmological applications of the present method are possible, but are not pursued here. Instead an indication is given of some of the consequences of adopting the simple model.

To illustrate the above, we shall give an example: At the late stages of cosmic expansion, it is commonly accepted that the matter can be reasonably represented as dust (i.e., a fluid with zero temperature and pressure). As shown by Hiscock and Salmonson,[7] one of the paradoxes which have characterized the Eckart and Landau–Lifshitz theories of relativistic fluids has been the prediction of effects violating the properties of dust. The nature of this problem lies in the fact that the value of the product of the mean time of relaxation υ^{-1} and Hubble's parameter H does not evolve toward zero, but, instead persists for all times and tends to infinity as $t \Rightarrow \infty$. Consequently, the well-known condition $\upsilon^{-1}H \ll 1$ under which the macroscopic fluid theories are consistent with kinetic theory is not satisfied, and these theories cannot be used to study the asymptotic behavior of a cosmological fluid.

Though inapplicable at the early stages of cosmic expansion, within its range of validity our direct method, which makes no assumptions concerning the value of $\upsilon^{-1}H$, provides a more credible description than the hydrodynamic theories of relativistic fluids. As an illustration, the discussion of Section 6 not only leads to a self-contained cosmological model, but also shows that the dust solution is entirely consistent with the large-time behavior of a Boltzmann gas. This in turn seems to give a better framework for understanding the relation between the Einstein–Boltzmann and Einstein–Liouville equations ($t \Rightarrow \infty$).

As to the physical interpretation of an assemblage of material particles, we may regard it as, e.g., a hydrogen gas during the matter-dominated epoch, since a redshift $Z \cong 1000$ until $Z \cong 30$ or so (see footnote 7). Whether a more realistic two- or three-fluid model is analytically tractable is, at present, unclear. However, one can easily generalize our method by including the dark matter effects (cf. Section 7).

Here we proceed as follows. In Section 2, we introduce the relativistic Boltzmann equation for the Robertson–Walker metric with flat spatial sections. In Sections 3 and 4, two different linearization procedures are described and asymptotic forms of the corresponding linearized Boltzmann equations are derived. The auxiliary technical material, being in fact a supplement to Section 3.3, is included as Appendices A and B. Section 5 provides explicit solutions of the asymptotic kinetic equations and then studies their elementary properties. In Sections 2–5, we assume that the expansion factor R in a $k=0$ Robertson–Walker metric is a given function of the time coordinate t. We also postulate that this function increases with increasing t and satisfies the condition

$$\lim_{t\Rightarrow\infty} R(t) = \infty \tag{1.1}$$

The problem of solving the full Einstein–Boltzmann system, which governs the temporal evolution of the distribution function f and the expansion factor R, is briefly discussed in Section 6. Section 7 is for final remarks.

2. PRELIMINARIES

For simplicity, we shall consider only a one-component classical relativistic gas of particles with rest mass $m \neq 0$ in a $k=0$ Robertson–Walker space-time,

$$g_{\mathrm{RW}} := -c^2\, dt \otimes dt + R^2(t)[dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3] \tag{2.1}$$

so it can be described by a singlet distribution function $f(t, p^k)$. The components of the particle four-momentum p with respect to a local orthonormal tetrad $\{c^{-1}\ \partial/\partial t, R^{-1}\ \partial/\partial x^k\}$ will be denoted by p^α. In what follows, Latin indices will range from 1 to 3, Greek indices from 0 to 3.

We postulate that f satisfies the Boltzmann equation, which in the present case takes the form[1,8,9]

$$\frac{\partial f}{\partial t} - Hp^k \frac{\partial f}{\partial p^k} = I(f, f) \tag{2.2}$$

where $I(f, f)$ is the collision term and H is Hubble's parameter:

$$H := \dot{R}/R \tag{2.3}$$

It is instructive to define the bilinear form

$$I(f, h) := \frac{c}{8p^0} \int \frac{1}{p_1^0}\, d^3p_1\, d\Omega\, gs\sigma(g, \Theta)(f'h_1' + f_1'h' - fh_1 - f_1 h) \tag{2.4}$$

in terms of which the collision term is $I(f, f)$. The notation is conventional. As abbreviations we use the symbol f_1 for $f(t, p_1^k)$, f' for $f(t, p'^k)$, and f_1' for $f(t, p_1'^k)$. Here p^α and p_1^α are particle four-momenta before collision which produce p'^α and $p_1'^\alpha$, respectively, after an encounter. The values of the relative momenta of the particles before and after collision

$$g^\alpha := p_1^\alpha - p^\alpha, \qquad g'^\alpha := p_1'^\alpha - p'^\alpha \tag{2.5}$$

are calculated from

$$g := (g^\alpha g_\alpha)^{1/2}, \qquad g' := (g'^\alpha g'_\alpha)^{1/2} = g \tag{2.6}$$

The scalar quantity s times c can be regarded as being the total microscopic energy in the center-of-mass frame:

$$s := (-s^\alpha s_\alpha)^{1/2} = (4m^2c^2 + g^2)^{1/2} \tag{2.7a}$$

$$s^\alpha := p^\alpha + p_1^\alpha = p'^\alpha + p_1'^\alpha = s'^\alpha \tag{2.7b}$$

As to the symbol $d\Omega$ in (2.4), this is an area element on the unit sphere in the three-space normal to s^α:

$$d\Omega := \sin(\Theta)\, d\Theta\, d\Psi, \qquad \Psi \in [0, 2\pi), \quad \Theta \in [0, \pi] \tag{2.8a}$$

$$\cos(\Theta) := (gg')^{-1}\, g'^\alpha g_\alpha = g^{-2} g'^\alpha g_\alpha \tag{2.8b}$$

Clearly, Θ defines the angle of scattering.

In order to complete the specification of the integral in (2.4), we must supply the scattering cross section σ observed in the center-of-mass frame as a given function of the collision invariants g and Θ. For concreteness sake, we assume that

$$\sigma(g, \Theta) = \sigma_0[(mc)^{-i-j}\,\sigma_1\, g^i + g^{-j}]\sin^q(\Theta) \tag{2.9}$$

where σ_0, σ_1, i, j, q are constants. Two cases can be distinguished. If $j > 0$, we set

$$\sigma_0 > 0, \qquad \sigma_1 > 0, \qquad -2 < q \leqslant 0 \tag{2.10a}$$

$$0 \leqslant i < q + 2, \qquad -1 - q < j \leqslant 1 \tag{2.10b}$$

If $j = 0$, we write

$$\sigma_1 = 0, \qquad \sigma_0 = 2r^2, \qquad q = 0 \tag{2.11}$$

and characterize the quantity r by saying that $2r$ is the diameter of the particle.

The theory based upon (2.9)–(2.11) corresponds to the so-called relativistic hard interactions and can be used to demonstrate the exponential decay and asymptotic approach to fluid dynamics of general solutions of the linearized Boltzmann equation. To the best of our knowledge, this was first shown by Dudyński and Ekiel-Jeżewska.[3] As a matter of fact, we impose here stronger restrictions on the class of possible scattering cross sections than those proposed in ref. 3, because some additional, specializing assumptions are necessary if the correct nonrelativistic limit of the relativistic kinetic theory is to follow.

3. LINEARIZATION ABOUT THE MAXWELLIAN DISTRIBUTION

3.1. Definition of a Maxwellian Molecular Density

The Maxwellian distribution function is given by

$$F(z) := \frac{n}{(2\pi m k_B T)^{3/2}} \exp(-z^2) \tag{3.1a}$$

where k_B is the Boltzmann constant and z is characterized by

$$z := (y^k y_k)^{1/2}, \qquad y^k := (2mk_B T)^{-1/2} p^k \tag{3.1b}$$

The parameters n and T depend in general on time. We refer to mn as the classical equilibrium mass density and to T as the classical equilibrium temperature. We assume further that F satisfies the Liouville equation

$$\frac{\partial f}{\partial t} - Hp^k \frac{\partial f}{\partial p^k} = 0 \tag{3.2}$$

From this assumption one can read off conditions that n and T be related to the expansion factor R by

$$n(t) := 4\pi\omega R^{-3}(t), \qquad \omega = \text{const} \tag{3.3a}$$

$$T(t) := wk_B^{-1} R^{-2}(t), \qquad w = \text{const} \tag{3.3b}$$

Combining (3.3a) and (3.3b) then yields

$$\frac{n}{(2\pi m k_B T)^{3/2}} = \left(\frac{2}{\pi}\right)^{1/2} \frac{\omega}{(mw)^{3/2}} = \text{const} \tag{3.4}$$

3.2. The Alternative Form of the Boltzmann Equation

It is always possible to define the perturbation with respect to F and to look for a solution of the Boltzmann equation in the form

$$f(t, p^k) = F(z)[1 + \psi(t, y^k)] \tag{3.5}$$

Using (3.5), one can now express Eq. (2.2) in terms of ψ only,

$$\partial_t \psi = \frac{1}{F} I(F, F) + \frac{2}{F} I(F, F\psi) + \frac{1}{F} I(F\psi, F\psi) \tag{3.6}$$

With the definitions

$$\kappa := \frac{k_{\mathrm{B}} T}{mc^2} \tag{3.7a}$$

$$y_1^k := (2mk_{\mathrm{B}} T)^{-1/2} p_1^k, \qquad z_1 := (y_1^k y_{1k})^{1/2} \tag{3.7b}$$

$$G := [(y_1^k - y^k)(y_{1k} - y_k)]^{1/2} \tag{3.7c}$$

$$g_* := \left\{ -\frac{1}{2\kappa} [(1 + 2\kappa z_1^2)^{1/2} - (1 + 2\kappa z^2)^{1/2}]^2 + G^2 \right\}^{1/2} \tag{3.7d}$$

$$\sigma_*(g_*, \Theta) := [\sigma_1 (2\kappa)^{(i+j)/2} g_*^i + g_*^{-j}] \sin^q(\Theta) \tag{3.7e}$$

we readily find

$$F^{-1}(F\psi, F\varphi) = \upsilon J_\kappa(\psi, \varphi) \tag{3.8a}$$

where

$$\upsilon := \frac{n\sigma_0}{4\pi^{3/2} m} (2mk_{\mathrm{B}} T)^{(1-j)/2} \tag{3.8b}$$

$$J_\kappa(\psi, \varphi) := \int d^3 y_1 \, d\Omega \, \exp(z^2) \, \sigma_* \mathscr{P}_\kappa \mathscr{R}_\kappa(\psi, \varphi) \tag{3.8c}$$

$$\mathscr{P}_\kappa := \left[\frac{1 + \frac{1}{2}\kappa g_*^2}{(1 + 2\kappa z^2)(1 + 2\kappa z_1^2)} \right]^{1/2} g_* \tag{3.8d}$$

$$\begin{aligned} \mathscr{R}_\kappa(\psi, \varphi) := {} & \exp[-(z')^2 - (z_1')^2](\psi' \varphi_1' + \psi_1' \varphi') \\ & - \exp[-(z)^2 - (z_1)^2](\psi \varphi_1 + \psi_1 \varphi) \end{aligned} \tag{3.8e}$$

$$z' := (y'^k y_k')^{1/2}, \qquad z_1' := (y_1'^k y_{1k}')^{1/2} \tag{3.8f}$$

$$y'^k := (2mk_{\mathrm{B}} T)^{-1/2} p'^k, \qquad y_1'^k := (2mk_{\mathrm{B}} T)^{-1/2} p_1'^k \tag{3.8g}$$

Hence

$$\partial_t \psi = \upsilon \{ J_\kappa(1, 1) + 2J_\kappa(1, \psi) + J_\kappa(\psi, \psi) \} \tag{3.9}$$

Note that (3.9) is exact and is equivalent to the original Boltzmann integrodifferential equation.

The inverse of υ deserves to be regarded as the classical effective time of relaxation. Substituting (3.3) into (3.8b), we conclude that υ depends on time through the expansion factor R. The same remark concerns the normalized temperature κ as defined by (3.7a). It is easy to verify that the condition $J_\kappa(1, 1) = 0$ does not hold in the relativistic kinetic theory except in some kind of limit in which κ becomes zero. Thus, none of the Maxwellian distribution functions can serve to give a solution to the relativistic Boltzmann equation. Clearly, this does not mean that we cannot linearize the Boltzmann equation about F. The only true problem one faces in that approach is to show that the resulting kinetic equation for ψ be consistent and soluble.

3.3. The Asymptotic Kinetic Equation Deduced from (3.9)

Because of the nonlinear nature of the collision term, the Boltzmann equation as defined by (2.2) or by (3.9) is very difficult to solve and to analyze. The linear approximation is a familiar mathematical technique. Here, if κ and ψ are small, it consists in linearizing $J_\kappa(\psi, \varphi)$ with respect to κ and in neglecting, in the expression on the rhs of (3.9), nonlinear terms in the perturbation variables κ, ψ. The relative simplification it introduces need hardly be stressed. Moreover, this simplification is valid for the late stages of cosmic expansion where the behavior appropriate to a non-relativistic gas is encountered. By way of digression, the well-known problem of gauge-invariant quantities does not appear here, because we regard R as a given function of time.

In order to linearize the Boltzmann equation with respect to κ and ψ, we must first establish the limiting dependence of $\kappa^{-1}J_\kappa(1, 1)$ and $J_\kappa(1, \psi)$ on κ as κ tends to zero. From the analysis of Appendices A and B it follows that

$$\lim_{\kappa \Rightarrow 0} [J_\kappa(1, \psi)] = -\tfrac{1}{2}L[\psi] \tag{3.10a}$$

$$\lim_{\kappa \Rightarrow 0} [\kappa^{-1}J_\kappa(1, 1)] = \tfrac{1}{2}L[Q] \tag{3.10b}$$

where

$$L[\psi] := 2\int d^3y_1\, d\Omega \exp(-z_1^2)\, G\sigma_N(G, \Theta) \times (\psi + \psi_1 - \psi' - \psi'_1)_{(N)} \tag{3.11a}$$

$$\sigma_N(G, \Theta) := G^{-j}\sin^q(\Theta) \tag{3.11b}$$

$$(\psi + \psi_1 - \psi' - \psi'_1)_{(N)} := \lim_{\kappa \Rightarrow 0} (\psi + \psi_1 - \psi' - \psi'_1) \tag{3.11c}$$

$$Q(z) := \tfrac{15}{4} - 5z^2 + z^4 \tag{3.11d}$$

Now, a glance at (3.11a)–(3.11c) shows that L is the classical collision operator,[6] as it should be. An important aspect of these calculations is that the dominant parts of $J_\kappa(1, 1)$ and $J_\kappa(1, \psi)$ are given by $\frac{1}{2}\kappa L[Q]$ and $-\frac{1}{2}L[\psi]$, respectively, and that they can be expressed in terms of the classical collision operator. In Appendices A and B, we outline only the proof of (3.10b). As to (3.10a), this is an obvious result valid for those perturbations which fall off sufficiently rapidly with increasing z.

To the required linear approximation, which makes no reference to physical arguments relating collision times to expansion times, Eq. (3.9) becomes

$$\partial_t \psi = v\{\tfrac{1}{2}\kappa L[Q] - L[\psi]\} \tag{3.12}$$

Our objective in this paper is to solve Eq. (3.12), which gives the desired information about the gas in the nonrelativistic range of temperatures, without solving the original Boltzmann equation, which would give the most complete information possible. Also, it seems that the analytical problem for any case other than that described here must be extremely difficult. Now, it is a straightforward matter to verify that $\psi = 0$ does not satisfy (3.12). This is just what we found previously in the case of (3.9). Thus our asymptotic equation differs substantially from the linearized Boltzmann equation as derived within the framework of a classical kinetic theory.

We now turn to a brief treatment of the source term appearing on the rhs of (3.12). The exact form of $L[Q]$ depends in a rather complicated way on j and q [cf. Eq. (3.11b)], and the most general expression for $L[Q]$ is clearly impossibly cumbersome. For simplicity, we consider only two representative cases.

"Maxwellian" particles ($j = 1$)

$$L[Q] = 2\pi^{5/2}\xi Q \tag{3.13a}$$

$$\xi := \int_0^\pi \sin^{q+3}(\Theta)\, d\Theta \tag{3.13b}$$

Hard-sphere model ($j = 0$, $q = 0$)

$$L[Q] = \pi^{5/2}\frac{1}{z}\operatorname{erf}(z)\left(13 - 2z^2 - 12z^4 + \frac{8}{3}z^6\right) + 8\pi^2\left(\frac{3}{4} - \frac{5}{3}z^2 + \frac{1}{3}z^4\right)\exp(-z^2) \tag{3.14a}$$

$$\operatorname{erf}(z) := 2\pi^{-1/2}\int_0^z \exp(-x^2)\, dx \tag{3.14b}$$

4. LINEARIZATION ABOUT THE MAXWELL–JÜTTNER DISTRIBUTION

4.1. Definition of the Equilibrium Phase Density

In the relativistic kinetic theory, we regard as being appropriate to equilibrium any molecular-density function $\mathbb{F}$ such as to be left unaltered by collisions:

$$I(\mathbb{F}, \mathbb{F}) = 0 \tag{4.1}$$

If we recall the conditions stated in the introduction, we conclude that any such $\mathbb{F}$ is of the form

$$\mathbb{F} := (2\pi\hbar)^{-3} \exp\left(\frac{\mu - cp^0}{k_B \vartheta}\right) \tag{4.2}$$

where $2\pi\hbar$ is Planck's constant and μ and ϑ are arbitrary functions of time. In what follows, we refer to $\mathbb{F}$ as the Maxwell–Jüttner distribution function.

Introducing the abbreviations [cf. also Eqs. (3.1b), (3.3), and (3.7a)]

$$A := \frac{(2\pi m k_B T)^{3/2}}{(2\pi\hbar)^3 n} \exp\left(\frac{\mu - mc^2}{k_B \vartheta}\right) \tag{4.3a}$$

$$B := T/\vartheta \tag{4.3b}$$

we find that the Maxwell–Jüttner distribution function becomes

$$\mathbb{F} = A \frac{n}{(2\pi m k_B T)^{3/2}} \exp\left[-\frac{2Bz^2}{1 + (1 + 2\kappa z^2)^{1/2}}\right] \tag{4.4}$$

For simplicity, we now postulate that A and B depend on time only through the normalized temperature κ,

$$A = A(\kappa), \qquad B = B(\kappa) \tag{4.5a}$$

and that

$$\lim_{\kappa \Rightarrow 0} [A(\kappa)] = 1, \qquad \lim_{\kappa \Rightarrow 0} [B(\kappa)] = 1 \tag{4.5b}$$

We also assume that A and B are differentiable with respect to κ ($\kappa > 0$). With a general choice of the functions A and B, we should find that these functions were not themselves expressed in terms of the expansion factor R alone, and we could not complete our program of making the linearization of (2.2) about $\mathbb{F}$ tractable.

4.2. Another Form of the Boltzmann Equation

We continue to denote the perturbation by ψ, but it is now defined by the statement that

$$f(t, p^k) = \mathbb{F}(t, z)[1 + \psi(t, y^k)] \tag{4.6}$$

The counterpart of Eq. (3.9) is then

$$\partial_t \psi = \kappa \gamma_\kappa (1 + \psi) + \upsilon A \{2 I_\kappa(1, \psi) + I_\kappa(\psi, \psi)\} \tag{4.7}$$

where

$$\gamma_\kappa(t, z) := 2A^{-1}A'H - \frac{4Hz^2}{1 + (1 + 2\kappa z^2)^{1/2}} \times \left\{B' - \frac{Bz^2}{(1 + 2\kappa z^2)^{1/2}\,[1 + (1 + 2\kappa z^2)^{1/2}]}\right\} \tag{4.8a}$$

$$A' := \frac{dA}{d\kappa}, \qquad B' := \frac{dB}{d\kappa} \tag{4.8b}$$

$$I_\kappa(\psi, \varphi) := (\upsilon A \mathbb{F})^{-1}\, I(\mathbb{F}\psi, \mathbb{F}\varphi) \tag{4.8c}$$

Concerning the definition of H and υ, see Eqs. (2.3) and (3.8b). As to the explicit definition of $I_\kappa(\psi, \varphi)$, it may be noted that if we set

$$z_* := z, z_1, z', z_1' \tag{4.9}$$

and replace $\exp(\pm z_*^2)$ by

$$\exp_\kappa(\pm z_*^2) := \exp\left[\pm \frac{2Bz_*^2}{1 + (1 + 2\kappa z_*^2)^{1/2}}\right] \tag{4.10}$$

in (3.8c) and (3.8e), then $I_\kappa(\psi, \varphi)$ is given by the expression on the rhs of (3.8c).

Since $\gamma_\kappa \sim H$, it is an easy matter to verify that $\psi = 0$ does not obey (4.7) when $H \neq 0$. Thus, just as in the case of F, the Maxwell–Jüttner distribution function for massive particles cannot be a solution of the Boltzmann equation in an expanding universe.[4] The calculations leading to (4.7) are exact, but there is an important difference between (3.9) and (4.7). In the first approach the function $\psi(t, y^k)$ does not depend on time when the effects of particle collisions can be neglected ($\sigma_0 = 0$), whereas in the second approach the unperturbed problem is characterized by

$$\partial_t \psi = \kappa \gamma_\kappa (1 + \psi) \tag{4.11}$$

The difference arises from the influential role played by the time-dependent functions A, B, and κ. Nevertheless, the general approach adopted in this section can follow much the same lines as those already made familiar, in that the complicated expression on the rhs of (4.7) is linearized with respect to κ and ψ, and we seek asymptotic solutions of interest for the late stages of cosmic expansion where the behavior appropriate to a nonrelativistic gas is encountered. It is only necessary to adjust suitably the functions A and B by the procedure described in Section 4.4, and this clearly establishes the link between both methods, though the description of the linearization procedure can be put most vividly in the first method for which the unperturbed problem is

$$\partial_t \psi = 0 \tag{4.12}$$

4.3. Solution of the Unperturbed Problem

In the absence of collisions, Eq. (4.11) gives information about the rate of change of ψ due to the fact that $H \neq 0$ $(H > 0)$. The solution of the unperturbed problem is

$$\psi(t, y^k) = \frac{\mathbb{F}(t_0, z)}{\mathbb{F}(t, z)} [1 + \psi(t_0, y^k)] - 1 \tag{4.13}$$

where $t \geqslant t_0$ and t_0 represents the initial time $(t_0 > 0)$. In order to evaluate the magnitude of ψ at time t, we define the norm $|\psi(t)|_{(t)}$ as follows:

$$|\psi(t)|_{(t)} := A\pi^{-3/2} \int \exp\left[-\frac{2Bz^2}{1 + (1 + 2\kappa z^2)^{1/2}}\right] |\psi(t, y^k)| \, d^3y \tag{4.14}$$

Consider, for simplicity, the case in which $A = 1$ and $B = 1$. Then an appeal to the well-known properties of the Bessel functions[5] yields

$$|\psi(t)|_{(t)} \leqslant 2\kappa_0(1 + 8\kappa_0 + 28\kappa_0^2 + 44\kappa_0^3 + 21\kappa_0^4) + |\psi(t_0)|_{(t_0)} \tag{4.15a}$$

where

$$\kappa_0 := \kappa(t_0) \tag{4.15b}$$

In obtaining (4.15a), we have made use of the fact that for $H > 0$ the normalized temperature κ is a decreasing function of time.

It appears from (4.15a) that $|\psi(t)|_{(t)} \ll 1$ if κ_0 has a value appreciably less than one and if the perturbation ψ is small initially. Also, let us observe that, although under certain circumstances the perturbation ψ is negligibly small for all times $(t \geqslant t_0)$, it does not actually approach zero as time goes on. This last conclusion can only be avoided by introducing the full Boltzmann equation and by considering a situation in which the rate of growth of the effective time of relaxation is no greater than that of the inverse of Hubble's parameter (cf. Section 5).

4.4. The Asymptotic Kinetic Equation Deduced from (4.7)

So far, the differentiable functions A and B satisfying (4.5b) may vary in any way. We now define ψ_E by

$$\psi_E := F^{-1}(\mathbb{F} - F) = A \exp\left\{z^2\left[1 - \frac{2B}{1+(1+2\kappa z^2)^{1/2}}\right]\right\} - 1 \tag{4.16}$$

and impose the condition that the precise form of A and B arises solely from the analysis of the relations

$$\langle 1, \psi_E \rangle = 0, \qquad \langle z^2, \psi_E \rangle = 0 \tag{4.17}$$

where

$$\langle \psi, \varphi \rangle := \pi^{-3/2} \int \exp(-z^2)\, \psi(y^k)\, \varphi(y^k)\, d^3y \tag{4.18}$$

The rigorous analysis of (4.17) is a cumbersome task, which cannot be performed on the level of generality that we are maintaining in this paper. Fortunately, the limiting case $\kappa \Rightarrow 0$ can be treated easily, and we obtain in addition to (4.5b) the following results:

$$\lim_{\kappa \Rightarrow 0} \{\kappa^{-1}[A(\kappa) - 1]\} = \lim_{\kappa \Rightarrow 0} [A'(\kappa)] = \tfrac{15}{8} \tag{4.19a}$$

$$\lim_{\kappa \Rightarrow 0} \{\kappa^{-1}[B(\kappa) - 1]\} = \lim_{\kappa \Rightarrow 0} [B'(\kappa)] = \tfrac{5}{2} \tag{4.19b}$$

Because of (4.8a), we then have[3]

$$\lim_{\kappa \Rightarrow 0} [\gamma_\kappa(t, z)] = H(t)\, Q(z) \tag{4.20}$$

where the function $Q(z)$ is defined by (3.11d). Moreover, as in (3.10a), we obtain

$$\lim_{\kappa \Rightarrow 0} [I_\kappa(1, \psi)] = -\tfrac{1}{2} L[\psi] \tag{4.21}$$

[3] In the nonrelativistic range of temperatures, $(p^k p_k)^{1/2}$ is about $(2mk_B T)^{1/2}$ for the vast majority of the particles. From (3.1b) we then conclude that the typical value of z is very much smaller than $(2\kappa)^{-1/2}$. Thus $2\kappa z^2 \ll 1$ and $\gamma_\kappa(t, z)$ is close to $H(t)\, Q(z)$.

L is the classical collision operator [cf. Eq. (3.11a)]. Clearly, $L[\psi]$ is a function of t and y^k.

Neglecting the terms which are nonlinear with respect to κ, ψ and adopting the approach of Section 3.3, we finally find that the asymptotic form of (4.7) is given by

$$\partial_t \psi = \kappa HQ - \upsilon L[\psi] \tag{4.22}$$

Beginning from (4.22), full agreement can be achieved with the results of the kinetic theory based upon (3.12) even though (3.12) and (4.22) are not identical. We briefly study these problems in Section 5.2. Throughout Section 5 we shall be interested only in the asymptotic kinetic equations (3.12) and (4.22) rather than the original nonlinear Boltzmann equation. This is because the most general treatment cannot be elementary, and because it is useful to consider first the simple problem of solving the asymptotic kinetic equations. Nevertheless, although one has no hint toward solving (2.2), the complete discussion of (2.2) is certainly required for the deepest understanding of the problem, and would be expected to be capable of yielding the asymptotic solutions in the low-temperature limit.

5. SOLUTION OF THE ASYMPTOTIC KINETIC EQUATIONS

5.1. Calculations According to the First Method

Equation (3.12) suggests the introduction of a Hilbert space $\mathscr{H}$ where the scalar product $\langle \psi, \varphi \rangle$ and the norm $\|\psi\|$ are defined by (4.18) and $(\langle \psi, \psi \rangle)^{1/2}$, respectively. For the scattering cross sections[4] $\sigma_N(G, \Theta) := G^{-j} \sin^q(\Theta)$ corresponding to the classical hard interactions,[10] we verify the existence of a positive constant λ such that

$$\langle \psi, L[\psi] \rangle \geqslant \lambda \, \|\psi\|^2 \tag{5.1}$$

for all functions ψ orthogonal to $\psi_* := 1, y^k, z^2$. We define the subspace $\mathscr{H}_o$ of $\mathscr{H}$ as follows:

$$\mathscr{H}_o := \{ \psi \in \mathscr{H} \colon \langle \psi, \psi_* \rangle = 0 \} \tag{5.2}$$

Let us assume that $\psi_0 := \psi(t_0)$ belongs to the domain of the operator

[4] Cf. Eqs. (2.9)–(2.11) as well as Eqs. (3.8b) and (3.11b).

L; we denote this domain by $\mathscr{D}(L)$. For each $\psi_0 \in \mathscr{D}(L)$, Eq. (3.12) has a unique solution given by

$$\psi(t) = \exp\left\{-\left[\int_{t_0}^{t} \upsilon(\sigma)\, d\sigma\right] L\right\} \psi_0 + \tfrac{1}{2}\int_{t_0}^{t} \upsilon(\sigma)\, \kappa(\sigma) \exp\left\{-\left[\int_{\sigma}^{t} \upsilon(s)\, ds\right] L\right\} L[Q]\, d\sigma \tag{5.3}$$

From (5.1) and (5.3) we conclude immediately that if $\psi_0 \in \mathscr{H}_o \cap \mathscr{D}(L)$, then $\psi(t) \in \mathscr{H}_o$ and the norm of $\psi(t)$ satisfies the inequality[5]

$$\|\psi(t)\| \leqslant X(t_0, t)\, \|\psi_0\| + Y(t_0, t)\, \|L[Q]\| \tag{5.4a}$$

where

$$X(t_0, t) := \exp\left[-\lambda \int_{t_0}^{t} \upsilon(\sigma)\, d\sigma\right] \tag{5.4b}$$

$$Y(t_0, t) := \tfrac{1}{2}\int_{t_0}^{t} \upsilon(\sigma)\, \kappa(\sigma) \exp\left[-\lambda \int_{\sigma}^{t} \upsilon(s)\, ds\right] d\sigma \tag{5.4c}$$

Integrating by parts, we obtain another useful expression for $Y(t_0, t)$:

$$Y(t_0, t) = \frac{1}{2\lambda}\left\{\kappa(t) - \kappa(t_0) \exp\left[-\lambda \int_{t_0}^{t} \upsilon(\sigma)\, d\sigma\right]\right\} - \frac{1}{2\lambda}\int_{t_0}^{t} \dot{\kappa}(\sigma) \exp\left[-\lambda \int_{\sigma}^{t} \upsilon(s)\, ds\right] d\sigma \tag{5.5}$$

Combining (3.3b) and (3.7a) yields

$$\dot{\kappa} = -2\kappa \dot{R}/R = -2\kappa H \tag{5.6}$$

For $H > 0$, the time derivative of κ is negative and $\kappa(t') < \kappa(t)$ when $t' > t$.

Consequences of (5.4) are

$$X(t_0, t) \leqslant 1 \qquad \text{when} \quad t \geqslant t_0 \tag{5.7a}$$

$$Y(t_0, t) \leqslant \frac{1}{2\lambda}\, \kappa(t_0) \qquad \text{when} \quad t \geqslant t_0 \tag{5.7b}$$

Clearly, $X(t_0, t)$ is a monotonically decreasing function of time. The inequalities (5.7) assert a kind of stability of the Maxwellian distribution

[5] In this context, we note that $L[Q]$ belongs to $\mathscr{H}_o \cap \mathscr{D}(L)$.

function. Indeed, if $H>0$ and $\psi_0 \in \mathscr{D}(L)$, the norm of the solution to (3.12) is bounded from above for all times,[6]

$$\|\psi(t)\| \leqslant \|\psi(t_0)\| + \frac{1}{2\lambda}\kappa(t_0)\,\|L[Q]\| \tag{5.8}$$

To proceed further, we now add the following postulate: There exists the time t_1, $t_1 > t_0$, such that if $t \in [t_0, t_1]$, the rate of growth of the effective time of relaxation $(\lambda v)^{-1}$ is no greater than that of the inverse of Hubble's parameter H,[7]

$$\lambda v(t) \geqslant H(t), \qquad t \in [t_0, t_1] \tag{5.9}$$

From (5.4b), (5.5), and (5.6) it is then evident that

$$X(t_0, t) \leqslant R(t_0)/R(t) \tag{5.10a}$$

$$Y(t_0, t) \leqslant \frac{1}{2\lambda}\kappa(t) + \frac{1}{\lambda}\kappa(t_0)\,R(t_0)/R(t) \tag{5.10b}$$

Consequently, if $t \in [t_0, t_1]$ and $\psi_0 \in H_o \cap \mathscr{D}(L)$, substitution of (5.10) in (5.4a) enables us to say that there is a trend in time: the norm $\|\psi(t)\|$ can be bounded by a function which decreases with increasing t. In the unrealistic but mathematically interesting case (see footnote 7) when the inequality $\lambda v(t) \geqslant H(t)$ holds for all times $(t \geqslant t_0)$, we have

$$\lim_{t \Rightarrow \infty} \|\psi(t)\| = 0 \tag{5.11}$$

Interpreting (5.11), we see that the inequality $\lambda v \geqslant H$ delivers a necessary and sufficient condition under which every small perturbation of F tends to zero as $t \Rightarrow \infty$.

5.2. Consistency Between (3.12) and (4.22)

In the second method, the definition of the perturbation ψ differs from that in the first method. The implication is that the asymptotic kinetic

[6] For the hard-sphere model, we find from (3.14) that $\|L[Q]\| = 8\pi^2(17\pi/6 + 63/4\sqrt{3})^{1/2}$. Fixing attention on this model, the application of the results of Pekeris *et al.*[11] yields $\lambda \geqslant 0.7339(8\pi^2)$. In the case of "Maxwellian" particles [cf. (3.13)], we have $\|L[Q]\| = 30^{1/2}\xi\pi^{5/2}$.

[7] Since the $k=0$ Robertson–Walker universe model expands asymptotically as $R \sim t^{2/3}$, there is an indication that for large enough t the inequality $\lambda v(t) \geqslant H(t)$ does not hold. However, we have verified that our postulate is reasonable and consistent over a large range of cosmic times. For example, (5.9) can be applied to study the time evolution of a hydrogen gas during the matter-dominated epoch since a redshift $Z \cong 1000$ until $Z \cong 30$ or so. The details of calculations are available on request.

equations (3.12) and (4.22) are not identical. Let φ be a solution of (3.9) and assume that ψ obeys (4.7). We say that φ and ψ are equivalent if

$$F(z)[1+\varphi(t, y^k)] = \mathbb{F}(t, z)[1+\psi(t, y^k)] \tag{5.12}$$

Equation (5.12) implies that

$$\psi = \varphi - \psi_E \tag{5.13}$$

where ψ_E is given by (4.16). Because of (4.17) and (4.19), we may calculate the dominant part of ψ_E. In the nonrelativistic range of z (see footnote 3), we have

$$\psi_E(t, z) = \tfrac{1}{2}\kappa(t)\, Q(z) + \cdots \tag{5.14}$$

Substituting $\psi = \varphi - \frac{1}{2}\kappa Q$ into (4.22), we immediately see from (5.6) that ψ satisfies (4.22) if and only if φ satisfies (3.12). Thus the kinetic theory based upon (3.12) is completely equivalent to that based upon (4.22). More specifically, use of (5.9) yields the conclusion that if $\psi(t_0) \in \mathscr{H}_o \cap \mathscr{D}(L)$, then the norm of the solution to (4.22) is bounded for $t \in [t_0, t_1]$ by a function which decreases with increasing t. When $\lambda v(t) \geqslant H(t)$ for all times $(t \geqslant t_0)$, the perturbation $\psi(t)$ tends to zero as $t \Rightarrow \infty$.

We summarize the above observations as follows. The same asymptotic behavior of f can be extracted from each of two linearization procedures, and in this sense a comparison is made of these approaches which reveals mutual agreement.

6. COMMENTS ON THE EINSTEIN–BOLTZMANN SYSTEM

The symmetry of the Robertson–Walker metric requires that the mass density ρ, the energy density ε, and the pressure[8] p are functions of the time coordinate t only. These functions must satisfy the following system of differential equations:[(7)]

$$\dot{\rho} + 3\rho H = 0 \tag{6.1a}$$

$$\dot{\varepsilon} + 3(\varepsilon + p)H = 0 \tag{6.1b}$$

$$\dot{H} + H^2 = -\frac{4\pi\mathscr{G}}{3c^2}(\varepsilon + 3p) \tag{6.1c}$$

$$H^2 = \frac{8\pi\mathscr{G}}{3c^2}\varepsilon \tag{6.1d}$$

[8] The symbol p here is not to be confused with the particle four-momentum also denoted by p.

In these equations, an overdot indicates differentiation with respect to time and $\mathcal{G}$ is Newton's gravitational constant.

Writing f in the form (3.5), as is always possible, it is only a matter of labor to prove that the functions ρ, ε, p are related to the functions[9] n, κ, ψ by

$$(mn)^{-1}\rho = 4\pi^{-1/2}\int_0^\infty z^2 \exp(-z^2)[1+\psi(t,z)]\,dz \tag{6.2a}$$

$$(mc^2n)^{-1}\varepsilon = 4\pi^{-1/2}\int_0^\infty z^2(1+2\kappa z^2)^{1/2} \exp(-z^2)[1+\psi(t,z)]\,dz \tag{6.2b}$$

$$(mc^2n)^{-1}p = \tfrac{8}{3}\pi^{-1/2}\kappa\int_0^\infty z^4(1+2\kappa z^2)^{-1/2} \exp(-z^2)[1+\psi(t,z)]\,dz \tag{6.2c}$$

For the purposes of this section we have replaced $\psi(t, y^k)$ by $\psi(t, z)$, since anisotropic perturbations are not in general consistent with the Robertson–Walker geometry, insofar as the full nonlinear Einstein–Boltzmann system is concerned.

We have to solve the exact Boltzmann equation (3.9) and Eq. (6.1d) simultaneously for ψ and R. If ψ and R satisfy (3.9) and (6.1d), the remaining cosmological equations in (6.1) are automatically fulfilled because of (6.2). Given these remarks, we conclude that the Einstein–Boltzmann system consists of (3.9) and (6.1d). The exact Einstein–Boltzmann system is very difficult to solve and to analyze. A procedure analogous to that of Section 3.3 is possible, however. If ψ and κ are small, we can linearize the "constitutive" expressions on the rhs of (6.2) with respect to ψ and κ, so obtaining

$$(mn)^{-1}\rho = 1 + \langle 1, \psi\rangle \tag{6.3a}$$

$$(mc^2n)^{-1}\varepsilon = 1 + \langle 1, \psi\rangle + \tfrac{3}{2}\kappa \tag{6.3b}$$

$$(mc^2n)^{-1}p = \kappa \tag{6.3c}$$

These results, which are consistent with the equations of state for a nonrelativistic gas, validate our procedure. Indeed, if ψ satisfies the asymptotic kinetic equation (3.12) and ρ, ε, p are given by (6.3), the time derivative of $\langle 1, \psi\rangle$ vanishes and the cosmological equations (6.1a) and (6.1b) are automatically fulfilled. In the low-temperature limit, the Einstein–Boltzmann system consists of (3.12), (6.1d), and (6.3b). This system can easily be solved, especially in the case when ψ is orthogonal to ψ_*. The

[9] We recall that n and κ are defined by (3.3) and (3.7a).

cosmological equation (6.1c) in turn is obtained by differentiating (6.1d) with respect to time and then using (6.1b). We now see that our asymptotic procedure is a tractable formulation permitting a self-consistent calculation of ψ and R from a knowledge of $\psi(t_0)$ and $R(t_0)$.

All these questions are, from the cosmological standpoint, questions about the large-time behavior of a Boltzmann gas. When so viewed, some of them can be answered very directly: The final state is dust. By this we mean that, for each value of m,[10] the $k = 0$ Robertson–Walker universe model expands asymptotically as $t^{2/3}$ and the temperature T evolves as in a nonrelativistic gas, $T \sim R^{-2}$. As explained already in ref. 7, the treatment of these problems by means of macroscopic models can lead to paradoxes (the "critical mass effect").

Another remark may also be in order. As the calculations giving (3.12) and (6.3) are valid for any expansion factor R satisfying (1.1), again the problem of gauge-invariant quantities does not appear here. Mathematically speaking, this means that we do not introduce the notion of a background solution for R:

$$R(t) = R_0(t)[1 + r(t)] \tag{6.4}$$

Instead, we obtain the dynamical equations directly for ψ and R.

One final word concerning the linearization procedure based upon (4.6). If we define the perturbation ψ by (4.6) rather than (3.5), the analysis of this section can be repeated essentially word for word with only slight technical changes in the method.

7. FINAL REMARKS

The method used to obtain a coupled system of dynamical equations for the evaluation of $\psi(t, z)$ and $R(t)$ may also be used, with slight extension, to describe the time evolution of $\psi(t, z)$ and $R(t)$ in the case when the spatial sections of a Robertson–Walker metric are no longer flat. Since a suitably normalized Maxwellian density, and in fact any differentiable function[11] of z, satisfies the Liouville equation, again an asymptotic kinetic equation is found to hold for small enough ψ and κ.

Moreover, it seems reasonable to consider the possibility that dark mass interacts only gravitationally with the Boltzmann gas. If the dark

[10] It is useful to observe that $m \Rightarrow \infty$ implies $v^{-1}H \Rightarrow \infty$ [cf. Eq. (3.8b)]. Consequently, in place of the Eckart model (which is valid when $v^{-1}H \ll 1$), an asymptotic theory of the Einstein–Boltzmann system may be of interest.

[11] If p^α are components of the particle four-momentum with respect to $\{c^{-1}\, \partial/\partial t, \partial/\partial x^k\}$, then z is defined by $z := (2mk_B T)^{-1/2}\,(p^k p_k)^{1/2}$.

mass is in weakly interacting particles (here understood to include massive neutrinos), then it can be described by a phase-space distribution function obeying the Liouville equation. This equation is coupled to the Boltzmann equation only through the expansion factor R and the Einstein field equations for its evaluation. Thus the system of asymptotic equations can be derived and solved as before.

Restricting attention to the Robertson–Walker universe models, the potential virtues of the present method come to the fore when it is desirable to verify the existence of a unique solution of the Cauchy problem. Indeed, the existence and the uniqueness of the isotropic solution to the asymptotic system (3.12), (6.1d), and (6.3b) can easily be demonstrated. The more general problem of solving the full Einstein–Boltzmann system still seems to be an open problem even in the relatively simple theory for which the metric is given by (2.1).

If one introduces an almost-Robertson–Walker model of the universe, the question of the stability of F or $\mathbb{F}$ arises. Interpreted from a slightly more general point of view, it would seem natural to extend the techniques developed in this paper to the investigation of various problems related to the large-scale structure of the universe. We hope to discuss all these questions in the future.

APPENDIX A. SOME USEFUL ASYMPTOTIC FORMULAS

We easily conclude from the mass-shell constraint that the normalized three-momenta of the particles after collisions, namely y'^k and $y_1'^k$, are completely described by giving y^k, y_1^k and Θ, Ψ, κ. A very convenient orthonormal triad to which the angles Θ, Ψ can be referred has been introduced by Israel (ref. 8, p. 1173). The detailed description of this triad, while elementary, is formally too elaborate to present here. Fortunately, the discussion may be considerably simplified by use of the exact but asymptotic expression for

$$(z')^2 + (z_1')^2 = y'^k y_k' + y_1'^k y_{1k}' \tag{A.1}$$

We specify this asymptotic expression as follows[12]:

Define $\mathscr{E}$ by

$$\mathscr{E} := \tfrac{1}{2} \lim_{\kappa \Rightarrow 0} (Q' + Q_1' - Q - Q_1) \tag{A.2}$$

[12] The details of the proof, which are purely technical and require the full machinery of a relativistic kinetic theory, are available on request.

and suppose that $z + z_1 < \kappa^{\delta}$, where δ is a constant satisfying $-\frac{1}{2} < \delta < 0$. Then $(z')^2 + (z'_1)^2$ is characterized by

$$(z')^2 + (z'_1)^2 = z^2 + z_1^2 + \kappa\mathscr{E} + \mathbb{X} \tag{A.3}$$

and there is a constant $\mathbb{A}_\delta$ (which depends only on δ) such that

$$|\mathbb{X}| \leqslant \mathbb{A}_\delta \kappa^{2+4\delta}(z^2 + z_1^2) \tag{A.4}$$

What meaning is to be attached to (A.3) and (A.4)? If $\mathscr{E}$ as given by (A.2) is expressed in terms of (y^k, y_1^k) and (Θ, Ψ), it is found from $z + z_1 < \kappa^{\delta}$ that

$$|\kappa\mathscr{E}| \leqslant \tfrac{5}{4}\kappa(z + z_1)^4 \leqslant \tfrac{5}{2}\kappa^{1+2\delta}(z^2 + z_1^2) \tag{A.5}$$

By combination of (A.4) and (A.5), one then shows that the dominant part of $(z')^2 + (z'_1)^2 - z^2 - z_1^2$ is $\kappa\mathscr{E}$ for $\kappa \Rightarrow 0$.

The above results when taken together with the inequality $|\exp(\xi) - 1 - \xi| \leqslant |\xi|^2 \exp(|\xi|)$ imply that

$$\exp[-(z')^2 - (z'_1)^2] = \exp(-z^2 - z_1^2)(1 - \kappa\mathscr{E} + \mathbb{Y}) \tag{A.6a}$$

where

$$z + z_1 < \kappa^{\delta}, \qquad -\tfrac{1}{4} < \delta < 0 \tag{A.6b}$$

$$|\mathbb{Y}| \leqslant \mathbb{B}_\delta \kappa^{2+6\delta} + \mathbb{C}_\delta \kappa^{2+8\delta} \tag{A.6c}$$

As regards $\mathbb{B}_\delta$ and $\mathbb{C}_\delta$, these are constants which depend only on δ.

APPENDIX B. A SKETCH OF THE PROOF OF (3.10b)

We can split $J_\kappa(1, 1)$ into two parts and then investigate these two parts separately. Making use of (3.8c), we have

$$J_\kappa(1, 1) = M_\kappa^{(\delta)} + N_\kappa^{(\delta)} \tag{B.1a}$$

where

$$-\tfrac{1}{8} < \delta < 0 \tag{B.1b}$$

$$M_\kappa^{(\delta)} := \int_{z + z_1 < \kappa^{\delta}} d^3y_1\, d\Omega \exp(z^2)\, \sigma_* \mathscr{P}_\kappa \mathscr{R}_\kappa(1, 1) \tag{B.1c}$$

$$N_\kappa^{(\delta)} := \int_{z + z_1 \geqslant \kappa^{\delta}} d^3y_1\, d\Omega \exp(z^2)\, \sigma_* \mathscr{P}_\kappa \mathscr{R}_\kappa(1, 1) \tag{B.1d}$$

For $z + z_1 < \kappa^\delta$, we apply the asymptotic formulas of Appendix A and the inequality[13]

$$\frac{G}{(1+2\kappa z^2)^{1/2}}\left[1+\left(\frac{\kappa}{2}\right)^{1/2} G\right]^{-1} \leqslant g_* \leqslant G \tag{B.2}$$

in order to prove that

$$\lim_{\kappa \Rightarrow 0} [\kappa^{-1} M_\kappa^{(\delta)}] = \tfrac{1}{2} L[Q] \tag{B.3}$$

For $z + z_1 \geqslant \kappa^\delta$ and sufficiently small values of κ, a simple calculation based upon

$$\begin{aligned}(z')^2 + (z_1')^2 &\geqslant \tfrac{1}{2}(z_1 - z)^2 + \tfrac{1}{2} g_*^2 \\ &\geqslant \tfrac{1}{2}(z_1 - z)^2 + z^2 \end{aligned} \tag{B.4}$$

shows that

$$\lim_{\kappa \Rightarrow 0} [\kappa^{-1} N_\kappa^{(\delta)}] = 0 \tag{B.5}$$

Hence a sketch of the proof of (3.10b) is complete.

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NOTE ADDED IN PROOF

At the late stages of cosmic expansion, two asymptotic procedures described here are not the only ones yielding *simple* results for the motion of a Boltzmann gas. Another approach, closer to geometrical concepts, is presented in refs. 12 and 13. It seems profitable to compare and contrast the predictions of these *different* theories. This will be done in a separate paper.

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